Meta-strategy for Learning Tuning Parameters with Guarantees

Dimitri Meunier (IIT) and Pierre Alquier (RIKEN AIP)

February 4, 2021

Abstract

Online gradient methods, like the online gradient algorithm (OGA), often depend on tuning parameters that are difficult to set in practice. We consider an online meta-learning scenario, and we propose a meta-strategy to learn these parameters from past tasks. Our strategy is based on the minimization of a regret bound. It allows to learn the initialization and the step size in OGA with guarantees. We provide a regret analysis of the strategy in the case of convex losses. It suggests that, when there are parameters $\theta_1, \dots, \theta_T$ solving well tasks $1, \dots, T$ respectively and that are close enough one to each other, our strategy indeed improves on learning each task in isolation.

1 Introduction

In many applications of modern supervised learning, such as medical imaging or robotics, a large number of tasks is available but many of them are associated with a small amount of data. With few datapoints per task, learning them in isolation would give poor results. In this paper, we consider the problem of learning from a (large) sequence of regression or classification tasks with small sample size. By exploiting their similarities we seek to design algorithms that can utilize previous experience to rapidly learn new skills or adapt to new environments.

Inspired by human ingenuity in solving new problems by leveraging prior experience, *meta-learning* is a subfield of machine learning whose goal is to automatically adapt a learning mechanism from past experiences to rapidly learn new tasks with little available data. Since it "learns the learning mechanism" it is also referred to as *learning-to-learn* [34]. It is seen as a critical problem for the future of machine learning [10]. Numerous formulations exist for meta-learning and we focus on the problem of *online meta-learning* where the tasks arrive one at a time and the goal is to efficiently transfer information from the previous tasks to the new ones such that we learn the new tasks as efficiently as possible (this has also been refered to as *lifelong learning*). Each task is in turn processed *online*. To sum up, we have a stream of tasks and for each task a stream of observations.

In order to solve online tasks, diverse well-established strategies exist: perceptron, online gradient algorithm, online mirror descent, follow-the-regularized-leader, exponentially weighted aggregation etc. We refer the reader to [8, 33, 18, 27] for introductions to these algorithms and to so-called regret bounds, that control their generalization errors. We refer to these algorithms the *within-task* strategies. The big challenge is to design a meta-strategy that uses past experiences to adapt a within-task strategy to perform better on the next tasks.

In this paper we propose a new meta-learning strategy. The main idea to learn the tuning parameters by minimizing its regret bound. We provide a meta-regret analysis for our strategy. We illustrate our results in the case where the within-task strategy is the online gradient algorithm (OGA). While previous works like [13] allowed to learn the starting point of gradient descent (initialization), our method also allow to learn the gradient step size, with guarantees. In this case, the regret with respect to using parameters $\theta_1, \dots, \theta_T$ for task $1, \dots, T$ respectively is in $(1 + \sigma(\theta_1^T)\sqrt{n})T$ where T is the number of tasks, n the number of steps in each task, and

$$\sigma^2(\theta_1^T) = \frac{1}{T} \sum_{t=1}^{T} \left\| \theta_t - \frac{1}{T} \sum_{s=1}^{T} \theta_s \right\|^2 . \tag{1.1}$$

On the other hand, learning in isolation leads to a regret in $T\sqrt{n}$. So, our meta-strategies improve on learning in isolation in the case where there are $\theta_1, \dots, \theta_T$ solving well task $1, \dots, T$ respectively, and such that $\sigma(\theta_1^T)$ is small.

1.1 Related works

Meta-learning is similar to multitask learning [23, 30, 35] in the sense that the learner faces many tasks to solve. However, in multitask learning, the learner is given a fixed number of tasks, and can learn the connections between these tasks. In meta-learning, the learner must prepare to face future tasks that are not given yet.

Meta-learning is often refered to as learning-to-learn or lifelong learning. [2] proposed the following distinction: "learning-to-learn" for situations where the tasks are presented simultaneously, and "lifelong learning" for situations where they are presented sequentially. Following this terminology, learning-to-learn algorithms were proposed very early in the literature, with generalization guarantees [6, 29, 24, 3, 31, 19].

On the other hand, in the lifelong learning scenario, until recently, algorithms were proposed without generalization guarantees [32, 4]. A theoretical study was proposed by [2], but the strategies in this paper are not feasible in practice. This problem was improved recently [11, 5, 12, 17, 36, 16, 14, 21]. The closest work to this paper is [13], where the authors propose an efficient strategy to learn the starting point of online gradient descent. However, an application of this strategy to learning the step size do not show any improvement over learning in isolation [25].

1.2 Organization of the paper

In Section 2, we introduce the formalism of meta-learning and the notations that will be used throughout the paper. In Section 3, we introduce our meta-learning strategy, and its theoretical analysis. In Section 4, we provide the details of our method in the case of meta-learning the initialization and the step size in the online gradient algorithm. Based on our theoretical results, we

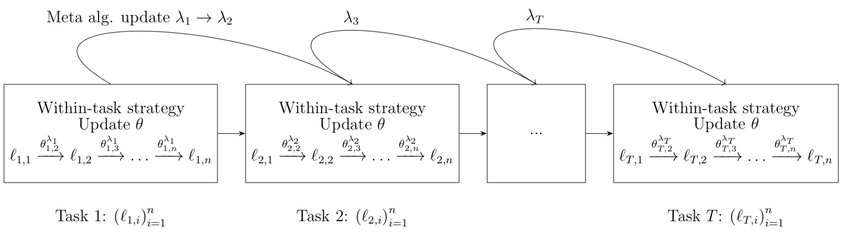

Figure 1: The dynamics of meta-learning.

also explicit situations where meta-learning indeed improves on learning the tasks independently. We also discuss how our strategy could be used to learn prior weights in Bayesian and aggregation methods. We report the results of an empirical evaluation in Section 5. The proofs of the main results are given in Section 6.

2 Notations and preliminaries

By convention, vectors $v \in \mathbb{R}^d$ are seen as $d \times 1$ matrices (columns). Let $\|v\|$ denote the Euclidean norm of v. Let A^T denote the transpose of any $d \times k$ matrix A. For two real numbers a and b, let $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. For $z \in \mathbb{R}$, z_+ is its positive part $z_+ = z \vee 0$.

The learner has to solve tasks $t = 1, \dots, T$ sequentially. Each task t consists in n rounds $i = 1, \dots, n$. At each round i of task t, the learner has to take a decision $\theta_{t,i}$ in a decision space $\Theta \subseteq \mathbb{R}^d$ for some $d > 0$. Then, a convex loss function $\ell_{t,i} : \Theta \to \mathbb{R}$ is revealed to the learner, who incurs the loss $\ell_{t,i}(\theta_{t,i})$. Classical examples with $\Theta \subset \mathbb{R}^d$ include regression tasks, where $\ell_{t,i}(\theta) = (y_{t,i} - x_{t,i}^T \theta)^2$ for some $x_{t,i} \in \mathbb{R}^d$ and $y_{t,i} \in \mathbb{R}$. For classification tasks, $\ell_{t,i}(\theta) = (1 - y_{t,i} x_{t,i}^T \theta)_+$ for some $x_{t,i} \in \mathbb{R}^d$, $y_{t,i} \in \{-1, +1\}$.

Throughout the paper, we will assume that the learner uses for each task an online decision strategy called *within-task strategy*, parametrized by a tuning parameter $\lambda \in \Lambda$ where Λ is a closed, convex subset of $\mathbb{R}^p$ for some $p > 0$. Example of such strategies include the online gradient algorithm (OGA), given by $\theta_{t,i} = \theta_{t,i-1} - \gamma \nabla \ell_{t,i}(\theta_{t,i-1})$. In this case, the tuning parameters are the initialization, or starting point, $\theta_{t,1} = \vartheta$ and the learning rate, or step size, γ. That is, $\lambda = (\vartheta, \gamma)$, so $p = d + 1$. The parameter λ is kept fixed during the whole task. It is of course possible to use the same parameter λ in *all* the tasks. However, we will be interested here in defining *meta-strategies* that will allow to improve λ task after task, based on the information available so far. In Section 3, we will define such strategies. For now, let λ_t denote the tuning parameter used by the learner all along task t. Figure 1 provides a recap of all the notations.

Let $\theta_{t,i}^\lambda$ denote the decision at round i of task t when the online strategy is used with parameter λ. We will assume that a regret bound is available for the within-task strategy. By this, we mean that the learner knows a function

$\mathcal{B}_n : \Theta \times \Lambda \to \mathbb{R}$ such that, for any task t, for any $\lambda \in \Lambda$,

$$\sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda}) \leq \underbrace{\inf_{\theta \in \Theta} \left\{ \sum_{i=1}^{n} \ell_{t,i}(\theta) + \mathcal{B}_n(\theta, \lambda) \right\}}_{=:\mathcal{L}_t(\lambda)}. \tag{2.1}$$

For OGA, regret bounds that can be found for example in [33, 18]. Other examples include exponentially weighted aggregation (EWA, bounds in [8]). More examples will be discussed in the paper. The quantity $\mathcal{B}_n(\theta, \lambda)$ is usually refered to as "the regret". We will call $\mathcal{L}_t(\lambda)$ the "meta-loss": it will be the criterion minimized by our meta-strategy.

The simplest meta-strategy is learning in isolation. That is, we keep $\lambda_t = \lambda_0 \in \Lambda$ for all the tasks. The total loss after task T is then given by:

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_0}) \leq \sum_{t=1}^{T} \mathcal{L}_t(\lambda_0). \tag{2.2}$$

However, when the learner uses a meta-strategy to really try to improve the tuning parameter at the end of each task, the total loss is given by $\sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_t})$. We will in this paper investigate strategies with meta-regret bounds, that is, bounds of the form

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_t}) \leq \inf_{\lambda \in \Lambda} \left\{ \sum_{t=1}^{T} \mathcal{L}_t(\lambda) + \mathcal{C}_T(\lambda) \right\}. \tag{2.3}$$

Of course, such bounds will be relevant only if the right-hand side of (2.3) is not larger than the r.h.s of (2.2), and is significantly smaller in some favorable settings. We show when this is the case in Section 4.

3 Meta-learning algorithms

In this section, we provide two meta-strategies to update λ at the end of each task. The first one is feasible only in the special case where we have an explicit formula for the (sub-)gradient of each $\mathcal{L}_t(\lambda)$. In Section 4, we provide an example where this is the case. The second meta-strategy can be used without this assumption. In both cases, we provide a regret bound as (2.3), under the following condition.

Assumption 3.1. *For any $t \in \{1, \dots, T\}$, the function $\lambda \mapsto \mathcal{L}_t(\lambda)$ is L-Lipschitz and convex.*

3.1 Special case: the gradient of the meta-loss is available in close form

As each $\mathcal{L}_t$ is convex, its subdifferential at each point of Λ is non-empty. For the sake of simplicity, we will use the notation $\lambda \mapsto \nabla \mathcal{L}_t(\lambda)$ in the following formulas to denote *any* element of its subdifferential at λ. We define the online gradient meta-strategy (OGMS) with step $\alpha > 0$ and starting point $\lambda_1 \in \Lambda$: for any $t > 1$,

$$\lambda_t = \Pi_\Lambda[\lambda_{t-1} - \alpha \nabla \mathcal{L}_{t-1}(\lambda_{t-1})] \tag{3.1}$$

where Π_Λ denotes the orthogonal projection on Λ.

3.2 The general case

We now cover the general case, where a formula for the gradient of $\mathcal{L}_t(\lambda)$ might not be available. We propose the online proximal meta-strategy (OPMS) with step $\alpha > 0$ and starting point $\lambda_1 \in \Lambda$, defined by:

$$\lambda_t = \operatorname*{argmin}_{\lambda \in \Lambda} \left\{ \mathcal{L}_{t-1}(\lambda) + \frac{\|\lambda - \lambda_{t-1}\|^2}{2\alpha} \right\}. \tag{3.2}$$

Using classical notations, e.g [28], we can rewrite this definition with the proximal operator (hence the name of the method). Indeed $\lambda_t = \mathrm{prox}_{\alpha \mathcal{L}_{t-1}}(\lambda_{t-1})$ where prox is the proximal operator given by, for any $x \in \Lambda$ and any convex function $f : \Lambda \to \mathbb{R}$,

$$\mathrm{prox}_f(x) = \operatorname*{argmin}_{\lambda \in \Lambda} \left\{ f(\lambda) + \frac{\|x - \lambda\|^2}{2} \right\}. \tag{3.3}$$

This strategy is feasible in practice in the regime we are interested in, that is, when n is small or moderately large, and $T \to \infty$. The learner has to store all the losses of the current task $\ell_{t-1,1}, \dots, \ell_{t-1,n}$. At the end of the task, the learner can use any convex optimization algorithm to minimize, with respect to $(\theta, \lambda) \in \Theta \times \Lambda$, the function

$$F_t(\theta, \lambda) = \sum_{i=1}^{n} \ell_{t,i}(\theta) + \mathcal{B}_n(\theta, \lambda) + \frac{\|\lambda - \lambda_{t-1}\|^2}{2\alpha}. \tag{3.4}$$

We can use a (projected) gradient descent on F_t or its accelerated variants [26].

3.3 Regret analysis

Proposition 3.1. *Under Assumption 3.1, using either OGMS or OPMS with step $\alpha > 0$ and starting point $\lambda_1 \in \Lambda$ leads to*

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_t}) \leq \inf_{\lambda \in \Lambda} \left\{ \sum_{t=1}^{T} \mathcal{L}_t(\lambda) + \frac{\alpha T L^2}{2} + \frac{\|\lambda - \lambda_1\|^2}{2\alpha} \right\}. \tag{3.5}$$

The proof can be found in Section 6.

4 An example: learning the tuning parameters of online gradient descent

4.1 Explicit meta-regret bound

In all this section, we work under the following condition.

Assumption 4.1. *For any $(t, i) \in \{1, \dots, T\} \times \{1, \dots, n\}$, the function $\ell_{t,i}$ is Γ-Lipschitz and convex.*

We study the situation where the learner uses (projected) OGA as a within-task strategy, that is $\Theta = \{\theta \in \mathbb{R}^d : \|\theta\| \leq C\}$ and, for any $i > 1$,

$$\theta_{t,i} = \Pi_\Theta[\theta_{t,i-1} - \gamma \nabla \ell_{t,i}(\theta_{t,i-1})]. \tag{4.1}$$

With such a strategy, we already mentioned that $\lambda = (\vartheta, \gamma) \in \Lambda \subset \Theta \times \mathbb{R}_+$ contains an initialization and a step size. An application of the results in Chapter 11 in [8] gives $\mathcal{B}_n(\theta, \lambda) = \mathcal{B}_n(\theta, (\vartheta, \gamma)) = \gamma \Gamma^2 n/2 + \|\theta - \vartheta\|^2/(2\gamma)$. So

$$\mathcal{L}_t((\vartheta, \gamma)) = \inf_{\|\theta\| \leq C} \left\{ \sum_{i=1}^{n} \ell_{t,i}(\theta) + \frac{\gamma \Gamma^2 n}{2} + \frac{\|\theta - \vartheta\|^2}{2\gamma} \right\}. \tag{4.2}$$

It is quite direct to check Assumption (3.1). We summarize this in the following proposition.

Proposition 4.1. *Under Assumption 4.1, assume that the learner uses OGA as an inner algorithm. Assume* $\Lambda = \{\vartheta \in \mathbb{R}^d : \|\vartheta\| \leq C\} \times [\underline{\gamma}, \bar{\gamma}]$ *for some* $C > 0$ *and* $0 < \underline{\gamma} < \bar{\gamma} < \infty$*. Then Assumption 3.1 is satisfied with*

$$L := \sqrt{\frac{n^2 \Gamma^4}{4} + \frac{4C^2}{\underline{\gamma}^2} + \frac{4C^4}{\underline{\gamma}^4}}. \tag{4.3}$$

So, when the learner uses one of the meta-strategies OGMS or OPMS, we can apply Proposition 3.1 respectively. This leads to the following theorem.

Theorem 4.2. *Under the assumptions of Proposition 4.1, with* $\underline{\gamma} = 1/n^{\beta}$ *for some* $\beta > 0$ *and* $\bar{\gamma} = C^2$*, when the learner uses either OGMS or OPMS with*

$$\alpha = \frac{C}{L} \sqrt{\frac{4 + C^2}{T}} \tag{4.4}$$

(where L is given by (4.3)*), we have:*

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_t}) \leq \inf_{\theta_1, \dots, \theta_T \in \Theta} \left\{ \sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_t) \right.$$
$$\left. + \mathcal{C}(\beta, \Gamma, C) \left[n^{1 \vee 2\beta} \sqrt{T} + \left(n^{1-\beta} + \sigma(\theta_1^T) \sqrt{n} \right) T \right] \right\} \tag{4.5}$$

where $\mathcal{C}(\beta, \Gamma, C) > 0$ *depends only on* (β, Γ, C) *and where:*

$$\sigma(\theta_1^T) = \sqrt{\frac{1}{T} \sum_{t=1}^{T} \left\| \theta_t - \frac{1}{T} \sum_{s=1}^{T} \theta_s \right\|^2}. \tag{4.6}$$

Let us compare this result with learning in isolation. For a γ in $1/\sqrt{n}$, OGA leads to a regret in $\sqrt{n}$. After T tasks, learning in isolation thus leads to a regret in $T\sqrt{n}$. Our strategies with $\beta = 1$ lead to a regret in

$$n^2 \sqrt{T} + \left(1 + \sigma(\theta_1^T) \sqrt{n}\right) T. \tag{4.7}$$

The term $n^2\sqrt{T}$ is the price to pay for meta-learning. In the regime we are interested in (small n, large T), it is smaller than $T\sqrt{n}$. Consider the leading term. In the worst case, it is also in $T\sqrt{n}$. However, when there are good predictors $\theta_1, \dots, \theta_T$ for tasks $1, \dots, T$ respectively such that $\sigma(\theta_1^T)$ is small, we see the improvement with respect to learning in isolation. The extreme case is when there is a good predictor θ^* that predicts well for all the tasks. In this case, the regret with respect to $\theta_1 = \cdots = \theta_T = \theta^*$ is in $n^2\sqrt{T} + T$, which improves significantly on learning in isolation.

Let us now discuss the implementation of our meta-stategy. We first remark that under the quadratic loss, it is possible to derive a formula for $\mathcal{L}_t$, which allows to use OGMS. We then discuss OPMS for the general case.

4.2 Special case: quadratic loss

First, consider $\ell_{t,i} = (y_{t,i} - x_{t,i}^T\theta)^2$ for some $y_{t,i} \in \mathbb{R}$ and $x_{t,i} \in \mathbb{R}^d$. Assumption 4.1 is satisfied if we assume moreover that all the $|y_{t,i}| \leq c$ and $\|x_{t,i}\| \leq b$, with $\Gamma = 2bc + 2b^2C$. In this case,

$$\mathcal{L}_t((\vartheta,\gamma)) = \inf_{\|\theta\|\leq C} \left\{ \sum_{i=1}^n (y_{t,i} - x_{t,i}^T\theta)^2 + \frac{\gamma\Gamma^2 n}{2} + \frac{\|\theta - \vartheta\|^2}{2\gamma} \right\}. \tag{4.8}$$

Define $Y_t = (y_{t,1}, \dots, y_{t,n})^T$ and $X_t = (x_{t,1}|\dots|x_{t,n})^T$. The minimizer of $\sum_{i=1}^n (y_{t,i} - x_{t,i}^T\theta)^2 + \|\theta - \vartheta\|^2/(2\gamma)$ with respect to θ is known as the ridge regression estimator:

$$\hat{\theta}_t = \left(X_t^T X_t + \frac{I}{2\gamma} \right)^{-1} \left(X_t^T Y_t + \frac{\vartheta}{2\gamma} \right). \tag{4.9}$$

It also coincides with the minimizer in the right-hand-side of (4.8) on the condition that $\|\hat{\theta}_t\| \leq C$. In this case, by pluging $\hat{\theta}_t$ in (4.8), we have a close form formula for $\mathcal{L}_t((\vartheta,\gamma))$, and an explicit (but cumbersome) formula for its gradient. It is thus possible to use the OGML strategy to update $\lambda = (\vartheta,\gamma)$.

4.3 The general case

In the general case, denote $\lambda_{t-1} = (\vartheta_{t-1}, \gamma_{t-1})$, then $\lambda_t = (\vartheta_t, \gamma_t)$ is obtained by minimizing

$$F_t(\theta,(\vartheta,\gamma)) = \sum_{i=1}^n \ell_{t,i}(\theta) + \frac{\gamma\Gamma^2 n}{2} + \frac{\|\theta - \vartheta\|^2}{2\gamma} + \frac{\|\vartheta - \vartheta_{t-1}\|^2 + (\gamma - \gamma_{t-1})^2}{2\alpha} \tag{4.10}$$

with respect to $\theta, \vartheta, \gamma$. Any efficient minimization procedure can be used. In our experiments, we used a projected gradient descent, the gradient being given by:

$$\frac{\partial F_t}{\partial \theta} = \sum_{i=1}^n \nabla \ell_{t,i}(\theta) + \frac{\theta - \vartheta}{\gamma}, \tag{4.11}$$

$$\frac{\partial F_t}{\partial \vartheta} = \frac{\vartheta - \theta}{\gamma} + \frac{\vartheta - \vartheta_{t-1}}{\alpha}, \tag{4.12}$$

$$\frac{\partial F_t}{\partial \gamma} = \frac{\Gamma^2 n}{2} - \frac{\|\theta - \vartheta\|^2}{2\gamma^2} + \frac{\gamma - \gamma_{t-1}}{\alpha}. \tag{4.13}$$

Note that even though we do not *stricto sensu* obtain the minimizer of F_t, we can get arbitrarily close to it by taking a large enough number of steps.

4.4 Learning priors in Bayesian inference

We now discuss a possible way to use OPMS to learn the prior and learning rate in generalized Bayesian inference or EWA. We say that the learner uses generalized Bayesian inference when:

$$\rho_{t,i}(\mathrm{d}\theta) = \operatorname*{argmin}_{\rho} \left\{ \mathbb{E}_{\theta\sim\rho}\left[\sum_{j=1}^{i-1} \ell_{t,j}(\theta)\right] + \frac{\mathcal{K}(\rho,\pi)}{\eta} \right\} \tag{4.14}$$

where the minimum is taken over all probability distributions absolutely continuous with π, π is a prior distribution, $\eta > 0$ a learning rate and $\mathcal{K}$ the Kullback-Leibler divergence (KL). Meta-learning for such an update rule is proven in [2] but usually does not lead to feasible strategies. Online variational inference [22, 9] consists in replacing the minimization on the set of all probability distributions by minimization in a smaller set in order to define a feasible approximation of $\rho_{t,i}$. For example, let $(q_\mu)_{\mu\in M}$ be a parametric family of probability distributions, we define:

$$\mu_{t,i} = \operatorname*{argmin}_{\mu\in M} \left\{ \mathbb{E}_{\theta\sim q_\mu}\left[\sum_{j=1}^{i-1} \ell_{t,j}(\theta)\right] + \frac{\mathcal{K}(q_\mu,\pi)}{\eta} \right\}. \tag{4.15}$$

It is discussed in [15] that generally, when μ is a location-scale parameter and $\ell_{t,j}$ is Γ-Lipschitz and convex, then $\bar{\ell}_{t,i}(\mu) := \mathbb{E}_{\theta\sim q_\mu}[\ell_{t,j}(\theta)]$ is 2Γ-Lipschitz and convex. In this case, under the assumption that $\mathcal{K}(q_\mu,\pi)$ is α-strongly convex in μ, a regret bound for such strategies was derived in [9]:

$$\sum_{i=1}^{n} \bar{\ell}_{t,i}(\mu_t) \leq \inf_{\mu\in\mathcal{M}} \left\{ \mathbb{E}_{\theta\sim q_\mu}\left[\sum_{i=1}^{n} \ell_{t,i}(\theta)\right] + \frac{\eta 4\Gamma^2 n}{\alpha} + \frac{\mathcal{K}(q_\mu,\pi)}{\eta} \right\}. \tag{4.16}$$

Assume $\mu = m \in \mathbb{R}^d$ and q_m is a Gaussian distribution with mean m and variance I, and assume that π is a Gaussian distribution with mean ϑ and variance I. We have:

$$\mathcal{K}(q_m,\pi) = \frac{\|m-\vartheta\|^2}{2}. \tag{4.17}$$

The convexity of $\ell_{t,i}$ ensures that $\mathbb{E}_{\theta\sim q_m}\left[\sum_{i=1}^n \bar{\ell}_{t,i}(\theta)\right] \leq \sum_{i=1}^n \ell_{t,i}[\mathbb{E}_{\theta\sim q_m}(\theta)] = \sum_{i=1}^n \ell_{t,i}(m)$. Pluging the upper bound into (4.15), we obtain:

$$m_{t,i} = \operatorname*{argmin}_{m\in\mathbb{R}^d} \left\{ \sum_{j=1}^{i-1} \ell_{t,i}(m) + \frac{\|m-\vartheta\|^2}{2\eta} \right\}, \tag{4.18}$$

that is, the FTRL strategy (Follow The Regularized Leader). OGA is obtained by a linearization of FTRL. Thus, OGA can be interpreted as an approximate Bayesian strategy where we only learn the mean of the best Gaussian approximation of the posterior. Moreover, OPMS allows to learn the prior π in this case, that is, ϑ (and we can also use it to learn the rate η).

More generally, when the KL divergence $\mathcal{K}(q_\mu,\pi)$ is not convex with respect to the parameters of π, we propose to replace it by a convex relaxation that would allow to use OGMS or OPMS. This relates to [20, 1] who advocate to go beyond the KL divergence in (4.14). This will be the object of a future work.

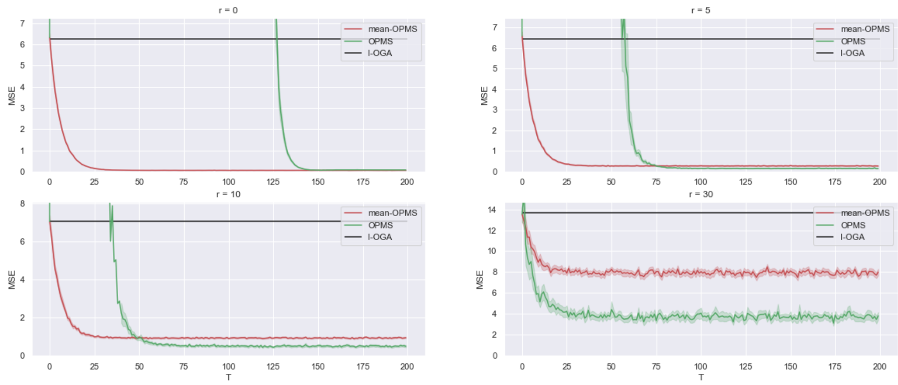

Figure 2: Performance of learning in isolation with OGA (**I-OGA**), OPMG to learn the initialization (**mean-OPMG**) and OPMG to learn the initialization and step size (**OPMG**). We report the sum of the MSE losses at the end of each task, for different values of the task-similarity index $r \in \{0, 5, 10, 30\}$. The results are averaged over 50 independent runs to get confidence intervals.

5 Experimental study

In this section we compare on simulated data the numerical performance of OPMS w.r.t learning the task in isolation with online gradient descent (I-OGA). To measure the impact of learning the gradient step γ, we also introduce mean-OPMS that uses the same strategy as OPMS but only learns the starting point ϑ (it is thus close to [13]). We present the results for regression tasks with the mean-squared-error loss. The notebook of the experiments can be found online: `https://dimitri-meunier.github.io/` We also provide a notebook for classification with the hinge loss, that led to similar results.

Synthetic Regression. At each round $t = 1, \ldots, T$, the meta learner receives sequentially a regression task that corresponds to a dataset $(x_{t,i}, y_{t,i})_{i=1,\ldots,n}$ generated as $y_{t,i} = x_{t,i}^T \theta_t + \epsilon_{t,i}$, $x_{t,i} \in \mathbb{R}^d$. The noise is $\epsilon_{t,i} \sim \mathcal{U}([-\sigma^2, \sigma^2])$, the inputs are uniformly sampled on the $(d-1)$-unit sphere $\mathcal{S}^{d-1}$ and $\theta_t = ru + \theta_0$, $u \sim \mathcal{U}\left(\mathcal{S}^{d-1}\right)$, $\theta_0 \in \mathbb{R}^d$, $r \in \mathbb{R}_+$. We take $d = 20$, $n = 30$, $T = 200$, $\sigma^2 = 0.5$ and θ_0 with all components equal to 5. In this setting, θ_0 is a common bias between the tasks, σ^2 is the inter-task variance and r characterizes the tasks similarity. We experiment with different values of $r \in \{0, 5, 10, 30\}$ to observe the impact of task similarity on the meta-learning process. The smaller r, the closer are the tasks and for the extreme case of $r = 0$ the tasks are identical. We draw attention to the fact that a cross-validation procedure to select α or λ is not valid in the online setting as it would require to know several tasks in advance for the former and several datapoints in advance for each task for the latter. Moreover, the theoretical values are based on worst-case analysis and lead in practice to slow learning. In practice, to set these values to the correct order of magnitude without adjusting the constants led to better results. So, for mean-OPMS and OPMS we set $\alpha = 1/\sqrt{T}$, for OPMS and I-OGA we set $\lambda = 1/\sqrt{n}$. Instead of cross-validation, one can launch several online learners in parallel with different parameters values to pick the best one (or aggregate them). That is the strategy we use to select Γ for OPMS. Note that the exact value of Γ is usually unkown in practice; its automatic calibration is an important open question. To solve (4.10), after each task we use the exact solution for mean-OPMS and projected Newton descent with 10 steps for OPMS. We observed that not reaching the exact solution of (4.10) does not harm the performance of the algorithm and 10 steps are sufficient to reach convergence. The results are displayed in Table 1 and Figure 2. On Figure 2, for each task $t = 1, \ldots, T$, we report the average end-of-task loss $MSE_t = \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,n})/n$ averaged over 50 independent runs (with their confidence intervals). Table 1 reports MSE_t averaged over the 100 last tasks. The results confirms our theoretical findings: learning γ can bring a substantial benefit over just learning the starting point, which in turn brings a considerable benefit with respect to learning the tasks in isolation. Learning the gradient step makes the meta-learner more robust to task dissimilarities (i.e. when r increases) as shown in Figure 2. In the regime where r is low, learning the gradient step does not help the meta-learner as it takes more steps to reach convergence. Overall both meta learners are consistently better than learning the task in isolation since the number of observation per task is low.

	r=0	r=5	r=10	r=30
I-OGA	6.24	6.44	7.06	13.60
mean OPMS	0.05	0.27	0.93	7.93
OPMS	0.07	0.15	0.49	3.72

Table 1: Average end-of-task MSE of the 100 last tasks (averaged over 50 independent runs).

6 Proofs

Lemma 6.1. *Let a, b, c be three vectors in $\mathbb{R}^p$. Then:*

$$(a-b)^T(b-c) = \frac{\|a-c\|^2 - \|a-b\|^2 - \|b-c\|^2}{2}. \tag{6.1}$$

Proof: expand $\|a-c\|^2 = \|a\|^2 + \|c\|^2 - 2a^T c$ in the r.h.s, as well as $\|a-b\|^2$ and $\|b-c\|^2$. Then simplify. $\square$

We now prove separately Proposition 3.1 for the general PGMS strategy, and then for OGMS.

Proof of Proposition 3.1 for OPMS: note that up to our knowledge, regret bounds for online updates based on the proximal operator were first studied in Exercice 11.3 in [8]. We here provide a detailed proof in our particular setting. Note that better bounds were recently proven in [7], where the order of the bound is improved under stronger assumptions.

First, λ_t is defined as the minimizer of a convex function in (3.1). So, the subdifferential of this function at λ_t contains 0. In other words, there is a $z_t \in \partial\mathcal{L}_{t-1}(\lambda_t)$ such that

$$z_t = \frac{\lambda_{t-1} - \lambda_t}{\alpha}. \tag{6.2}$$

By convexity, for any λ, for any $z \in \partial \mathcal{L}_{t-1}(\lambda_t)$,

$$\mathcal{L}_{t-1}(\lambda) \geq \mathcal{L}_{t-1}(\lambda_t) + (\lambda - \lambda_t)^T z. \tag{6.3}$$

The choice $z = z_t$ gives:

$$\mathcal{L}_{t-1}(\lambda) \geq \mathcal{L}_{t-1}(\lambda_t) + \frac{(\lambda - \lambda_t)^T (\lambda_{t-1} - \lambda_t)}{\alpha}, \tag{6.4}$$

that is,

$$\begin{aligned}\mathcal{L}_{t-1}(\lambda_t) &\leq \mathcal{L}_{t-1}(\lambda) + \frac{(\lambda - \lambda_t)^T (\lambda_t - \lambda_{t-1})}{\alpha} \\ &= \mathcal{L}_{t-1}(\lambda) + \frac{\|\lambda - \lambda_{t-1}\|^2 - \|\lambda - \lambda_t\|^2}{2\alpha} - \frac{\|\lambda_t - \lambda_{t-1}\|^2}{2\alpha} \\ &= \mathcal{L}_{t-1}(\lambda) + \frac{\|\lambda - \lambda_{t-1}\|^2 - \|\lambda - \lambda_t\|^2}{2\alpha} - \alpha \frac{\|z_t\|^2}{2} \end{aligned} \tag{6.5}$$

where we used Lemma 6.1. Then, note that

$$\begin{aligned}\mathcal{L}_{t-1}(\lambda_{t-1}) &= \mathcal{L}_{t-1}(\lambda_t) + [\mathcal{L}_{t-1}(\lambda_{t-1}) - \mathcal{L}_{t-1}(\lambda_t)] \\ &\leq \mathcal{L}_{t-1}(\lambda_t) + \|\lambda_{t-1} - \lambda_t\| L \\ &\leq \mathcal{L}_{t-1}(\lambda_t) + \alpha \|z_t\| L. \end{aligned} \tag{6.6}$$

Combine this inequality with (6.5) gives

$$\begin{aligned}\mathcal{L}_{t-1}(\lambda_{t-1}) \leq \mathcal{L}_{t-1}(\lambda) + \frac{\|\lambda - \lambda_{t-1}\|^2 - \|\lambda - \lambda_t\|^2}{2\alpha} \\ + \alpha \left(\|z_t\| L - \frac{\|z_t\|^2}{2} \right). \end{aligned} \tag{6.7}$$

Now, for any $x \in \mathbb{R}$, $-x^2/2 + xL - L^2/2 \leq 0$. In particular, $\|z_t\| L - \|z_t\|^2/2 \leq L^2/2$ and so the above can be rewritten:

$$\mathcal{L}_{t-1}(\lambda_{t-1}) \leq \mathcal{L}_{t-1}(\lambda) + \frac{\|\lambda - \lambda_{t-1}\|^2 - \|\lambda - \lambda_t\|^2}{2\alpha} + \frac{\alpha L^2}{2}. \tag{6.8}$$

Summing the inequality for $t = 2$ to $T + 1$ leads to:

$$\sum_{t=1}^{T} \mathcal{L}_t(\lambda_t) \leq \sum_{t=1}^{T} \mathcal{L}_t(\lambda) + \frac{\|\lambda - \lambda_1\|^2 - \|\lambda - \lambda_{T+1}\|^2}{2\alpha} + \frac{\alpha T L^2}{2}. \tag{6.9}$$

This ends the proof. □

Proof of Proposition 3.1 for OGMS: the beginning of the proof follows the proof of Theorem 11.1 in [8].

Note that we can rewrite (3.1) as

$$\begin{cases} \tilde{\lambda}_t = \lambda_{t-1} - \alpha \nabla \mathcal{L}_{t-1}(\lambda_{t-1}) \\ \lambda_t = \Pi_\Lambda(\tilde{\lambda}_t) \end{cases}$$

Rearranging the first line, we obtain:

$$\nabla \mathcal{L}_{t-1}(\lambda_{t-1}) = \frac{\lambda_{t-1} - \tilde{\lambda}_t}{\alpha}. \tag{6.10}$$

By convexity, for any λ,

$$\mathcal{L}_{t-1}(\lambda) \tag{6.11}$$

$$\geq \mathcal{L}_{t-1}(\lambda_{t-1}) + (\lambda - \lambda_{t-1})^T \nabla \mathcal{L}_{t-1}(\lambda_{t-1}) \tag{6.12}$$

$$= \mathcal{L}_{t-1}(\lambda_{t-1}) + \frac{(\lambda - \lambda_{t-1})^T(\lambda_{t-1} - \tilde{\lambda}_t)}{\alpha}, \tag{6.13}$$

that is,

$$\mathcal{L}_{t-1}(\lambda_{t-1}) \leq \mathcal{L}_{t-1}(\lambda) - \frac{(\lambda - \lambda_{t-1})^T(\lambda_{t-1} - \tilde{\lambda}_t)}{\alpha}. \tag{6.14}$$

Lemma 6.1 gives:

$$(\lambda - \lambda_{t-1})^T(\lambda_{t-1} - \tilde{\lambda}_t)$$

$$= \frac{\|\lambda - \tilde{\lambda}_t\|^2 - \|\lambda - \lambda_{t-1}\|^2 - \|\lambda_{t-1} - \tilde{\lambda}_t\|^2}{2}$$

$$= \frac{\|\lambda - \tilde{\lambda}_t\|^2 - \|\lambda - \lambda_{t-1}\|^2 - \alpha^2\|\nabla \mathcal{L}_{t-1}(\lambda_{t-1})\|^2}{2} \tag{6.15}$$

$$\geq \frac{\|\lambda - \lambda_t\|^2 - \|\lambda - \lambda_{t-1}\|^2 - \alpha^2\|\nabla \mathcal{L}_{t-1}(\lambda_{t-1})\|^2}{2}, \tag{6.16}$$

the last step being justified by:

$$\|\lambda - \tilde{\lambda}_t\|^2 \geq \|\lambda - \Pi_\Lambda(\tilde{\lambda}_t)\|^2 = \|\lambda - \lambda_t\|^2 \tag{6.17}$$

for any $\lambda \in \Lambda$. Plug (6.16) in (6.14) to get:

$$\mathcal{L}_{t-1}(\lambda_{t-1}) \leq \mathcal{L}_{t-1}(\lambda) + \frac{\|\lambda - \lambda_{t-1}\|^2 - \|\lambda - \lambda_t\|^2}{2\alpha} + \frac{\alpha\|\nabla \mathcal{L}_{t-1}(\lambda_{t-1})\|^2}{2} \tag{6.18}$$

and the Lipschitz assumption gives:

$$\mathcal{L}_{t-1}(\lambda_{t-1}) \leq \mathcal{L}_{t-1}(\lambda) + \frac{\|\lambda - \lambda_{t-1}\|^2 - \|\lambda - \lambda_t\|^2}{2\alpha} + \frac{\alpha L^2}{2} \tag{6.19}$$

Sum the inequality for $t = 2$ to $T + 1$ to get:

$$\sum_{t=1}^{T} \mathcal{L}_t(\lambda_t) \leq \sum_{t=1}^{T} \mathcal{L}_t(\lambda) + \frac{\|\lambda - \lambda_1\|^2 - \|\lambda - \lambda_{T+1}\|^2}{2\alpha} + \frac{\alpha T L^2}{2}. \tag{6.20}$$

This ends the proof of the statement for OGMS. □

Lemma 6.2. *Let $G(u, v)$ be a convex function of $(u, v) \in U \times V$. Define $g(u) = \inf_{v \in V} G(u, v)$. Then g is convex.*

Proof: indeed, let $\lambda \in [0, 1]$ and $(x, y) \in U^2$,

$$g(\lambda x + (1 - \lambda)y) \tag{6.21}$$

$$= \inf_{v \in V} G(\lambda x + (1-\lambda)y, v) \tag{6.22}$$
$$\leq G(\lambda x + (1-\lambda)y, \lambda x' + (1-\lambda)y') \tag{6.23}$$
$$\leq \lambda G(x, x') + (1-\lambda) G(y, y') \tag{6.24}$$

where the last two inequalities hold for any $(x', y') \in V^2$. Let us now take the infimum with respect to $(x', y') \in V^2$ in both sides, this gives:

$$g(\lambda x + (1-\lambda)y) \tag{6.25}$$
$$\leq \inf_{x' \in V} \lambda G(x, x') + \inf_{y' \in V} (1-\lambda) G(y, y') \tag{6.26}$$
$$= \lambda g(x) + (1-\lambda) g(y), \tag{6.27}$$

that is, g is convex. □
Proof of Proposition 4.1: apply Lemma 6.2 to $u = (\vartheta, \gamma)$, $v = \theta$, $U = \Lambda$, $V = \Theta$ and

$$G(u, v) = \sum_{i=1}^{n} \ell_{i,t}(\theta) + \frac{\gamma \Gamma^2 n}{2} + \frac{\|\vartheta - \theta\|^2}{2\gamma}. \tag{6.28}$$

This shows $g(u) = \mathcal{L}_t((\vartheta, \gamma))$ is convex with respect (ϑ, γ). Also, G is differentiable w.r.t $u = (\vartheta, \gamma)$, so

$$\frac{\partial G}{\partial \vartheta} = \frac{\vartheta - \theta}{\gamma}, \text{ and } \frac{\partial G}{\partial \gamma} = \frac{n\Gamma^2}{2} - \frac{\|\vartheta - \theta\|^2}{2\gamma^2}. \tag{6.29}$$

As a consequence, for $(\theta, \vartheta) \in \Theta^2$ and $\underline{\gamma} \leq \gamma \leq \overline{\gamma}$,

$$\left\| \frac{\partial G}{\partial \vartheta} \right\|^2 \leq \frac{4C^2}{\underline{\gamma}^2}, \text{ and } \left| \frac{\partial G}{\partial \gamma} \right|^2 \leq \frac{n^2 \Gamma^4}{4} + \frac{4C^4}{\underline{\gamma}^4}. \tag{6.30}$$

This leads to

$$\|\nabla_u G(u, v)\| = \sqrt{\left\| \frac{\partial G}{\partial \vartheta} \right\|^2 + \left| \frac{\partial G}{\partial \gamma} \right|^2} \tag{6.31}$$
$$= \sqrt{\frac{n^2 \Gamma^4}{4} + \frac{4C^2}{\underline{\gamma}^2} + \frac{4C^4}{\underline{\gamma}^4}} =: L, \tag{6.32}$$

that is, for each v, $G(u, v)$ is L-Lipschitz in u. So $g(u) = \inf_{v \in V} G(u, v)$ is L-Lipschitz in u. □
Proof of Theorem 4.2: thanks to the Assumption 4.1, we can apply Proposition 4.1. That is, Assumption (3.1) is satisfied, and we can apply Proposition 3.1. This gives:

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_t}) \leq \inf_{\theta_1, \dots, \theta_T \in \Theta} \inf_{(\vartheta, \gamma) \in \Lambda} \left\{ \sum_{t=1}^{T} \left[\sum_{i=1}^{n} \ell_{t,i}(\theta_t) + \frac{\gamma \Gamma^2 n}{2} + \frac{\|\theta_t - \vartheta\|^2}{2\gamma} \right] + \frac{\alpha T L^2}{2} + \frac{\|\vartheta - \vartheta_1\|^2 + |\gamma - \gamma_1|^2}{2\alpha} \right\}. \tag{6.33}$$

We use direct bounds for the last two terms: $\|\vartheta - \vartheta_1\|^2 \leq 4C^2$ and $|\gamma - \gamma_1|^2 \leq |\overline{\gamma} - \underline{\gamma}|^2 \leq \overline{\gamma}^2 = C^4$. Then note that

$$\sum_{t=1}^{T} \|\theta_t - \vartheta\|^2 = T \left\| \vartheta - \frac{1}{T} \sum_{s=1}^{T} \theta_s \right\|^2 + \sum_{t=1}^{T} \left\| \theta_t - \frac{1}{T} \sum_{s=1}^{T} \theta_s \right\|^2 \tag{6.34}$$

$$= T \left\| \vartheta - \frac{1}{T} \sum_{s=1}^{T} \theta_s \right\|^2 + T\sigma^2(\theta_1^T). \tag{6.35}$$

Upper bounding the infimum on ϑ in (6.33) by $\vartheta = \frac{1}{T}\sum_{s=1}^{T} \theta_s$ leads to

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_t}) \leq \inf_{\theta_1,\dots,\theta_T \in \Theta} \inf_{\gamma \in [\underline{\gamma}, \overline{\gamma}]} \left\{ \sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_t) + \frac{\gamma \Gamma^2 n T}{2} + \frac{T\sigma^2(\theta_1^T)}{2\gamma} + \frac{\alpha T L^2}{2} + \frac{C^2(4 + C^2)}{2\alpha} \right\}. \tag{6.36}$$

The right-hand side of (6.36) is minimized with respect to α if $\alpha = \frac{C}{L}\sqrt{\frac{4+C^2}{T}}$, which is the value proposed in the theorem, and we obtain:

$$\sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_t}) \leq \inf_{\theta_1,\dots,\theta_T \in \Theta} \inf_{\gamma \in [\underline{\gamma}, \overline{\gamma}]} \left\{ \sum_{t=1}^{T} \sum_{i=1}^{n} \ell_{t,i}(\theta_t) + \frac{\gamma \Gamma^2 n T}{2} + \frac{T\sigma^2(\theta_1^T)}{2\gamma} + CL\sqrt{(4 + C^2)T} \right\}. \tag{6.37}$$

The infimum with respect to γ in the r.h.s is reached for

$$\gamma^* = \left(\underline{\gamma} \vee \frac{\sigma(\theta_1^T)}{\Gamma\sqrt{n}} \right) \wedge \overline{\gamma}. \tag{6.38}$$

First, note that

$$\frac{\gamma^* \Gamma^2 n T}{2} \leq \left(\underline{\gamma} \vee \frac{\sigma(\theta_1^T)}{\Gamma\sqrt{n}} \right) \frac{\Gamma^2 n T}{2} \tag{6.39}$$

$$\leq \left(\underline{\gamma} + \frac{\sigma(\theta_1^T)}{\Gamma\sqrt{n}} \right) \frac{\Gamma^2 n T}{2} \tag{6.40}$$

$$= \frac{\Gamma^2 T n^{1-\beta}}{2} + \frac{\sigma(\theta_1^T)\Gamma T\sqrt{n}}{2}, \tag{6.41}$$

using $\underline{\gamma} = n^{-\beta}$. Then,

$$\frac{T\sigma^2(\theta_1^T)}{2\gamma^*} \leq \frac{T\sigma^2(\theta_1^T)}{2} \left(\frac{1}{\overline{\gamma}} \vee \frac{\Gamma\sqrt{n}}{\sigma(\theta_1^T)} \right) \tag{6.42}$$

$$\leq \frac{T\sigma^2(\theta_1^T)}{2} \left(\frac{1}{\overline{\gamma}} + \frac{\Gamma\sqrt{n}}{\sigma(\theta_1^T)} \right) \tag{6.43}$$

$$= \frac{T\sigma^2(\theta_1^T)}{2C^2} + \frac{\sigma(\theta_1^T)\Gamma T\sqrt{n}}{2} \tag{6.44}$$

$$\leq \frac{T\sigma(\theta_1^T)}{C} + \frac{\sigma(\theta_1^T)\Gamma T\sqrt{n}}{2}, \tag{6.45}$$

using $\overline{\gamma} = C^2$ and $\sigma(\theta_1^T) \leq 2C$. Pluging (6.39), (6.42) and the definition of L into (6.37) gives

$$\sum_{t=1}^{T}\sum_{i=1}^{n} \ell_{t,i}(\theta_{t,i}^{\lambda_t}) \leq \inf_{\theta_1,\dots,\theta_T\in\Theta}\left\{\sum_{t=1}^{T}\sum_{i=1}^{n}\ell_{t,i}(\theta_t) \right.$$
$$+ C\sqrt{\left(\frac{n^2\Gamma^4}{4} + 4C^2n^{2\beta} + 4C^4n^{4\beta}\right)(4+C^2)T}$$
$$\left. + \frac{\Gamma^2 T n^{1-\beta}}{2} + \sigma(\theta_1^T)T\left(\Gamma\sqrt{n} + \frac{1}{C}\right)\right\}. \tag{6.46}$$

This ends the proof. □

7 Conclusion

We proposed two simple meta-learning strategies together with their theoretical analysis. Our results clearly show an improvement on learning in isolation if the tasks are similar enough. These theoretical findings are confirmed by our numerical experiments. Important questions remain open. In [13], a purely online method is proposed, in the sense that it does not require to store all the information of the current task. In the case of OGA, this method allows to learn the starting point. However, its application to learn the step size is not direct [25]. An important question is then: is there a purely online method that would provably improve on learning in isolation in this case? Equally important open questions are the automatic calibration of Γ and the application to learn the prior π and η in generalized Bayesian inference (4.14).

Acknowledgements

This project was initiated as Dimitri Meunier's internship project at RIKEN AIP, in the Approximate Bayesian Inference team. The internship was cancelled because of the pandemics. We would like to thank Arnak Dalalyan (ENSAE Paris), who provided fundings so that the internship could take place at ENSAE Paris instead. We would like to thank Emtiyaz Khan (RIKEN AIP), Sébastien Gerchinovitz (IRT Saint-Exupéry, Toulouse), Vianney Perchet (ENSAE Paris) and all the members of the Approximate Bayesian Inference team for valuable feedback.

References

[1] Alquier, P. Non-exponentially weighted aggregation: regret bounds for unbounded loss functions. *arXiv preprint arXiv:2009.03017*, 2020.

[2] Alquier, P., Mai, T. T., and Pontil, M. Regret Bounds for Lifelong Learning. In Singh, A. and Zhu, J. (eds.), *Proceedings of the 20th International*

Conference on Artificial Intelligence and Statistics, volume 54 of *Proceedings of Machine Learning Research*, pp. 261–269, 2017.

[3] Amit, R. and Meir, R. Meta-learning by adjusting priors based on extended PAC-Bayes theory. In *International Conference on Machine Learning*, pp. 205–214, 2018.

[4] Andrychowicz, M., Denil, M., Gomez, S., Hoffman, M. W., Pfau, D., Schaul, T., Shillingford, B., and De Freitas, N. Learning to learn by gradient descent by gradient descent. In *Advances in neural information processing systems*, pp. 3981–3989, 2016.

[5] Balcan, M.-F., Khodak, M., and Talwalkar, A. Provable guarantees for gradient-based meta-learning. In *International Conference on Machine Learning*, pp. 424–433. PMLR, 2019.

[6] Baxter, J. Theoretical models of learning to learn. In *Learning to learn*, pp. 71–94. Springer, 1998.

[7] Campolongo, N. and Orabona, F. Temporal variability in implicit online learning. *Advances in Neural Information Processing Systems*, 33, 2020.

[8] Cesa-Bianchi, N. and Lugosi, G. *Prediction, learning, and games.* Cambridge University Press, 2006.

[9] Chérief-Abdellatif, B.-E., Alquier, P., and Khan, M. E. A generalization bound for online variational inference. *Proceedings of The Eleventh Asian Conference on Machine Learning, PMLR*, 101:662–677, 2019.

[10] Chollet, F. On the measure of intelligence. *arXiv preprint arXiv:1911.01547*, 2019.

[11] Denevi, G., Ciliberto, C., Stamos, D., and Pontil, M. Learning to learn around a common mean. In *Advances in Neural Information Processing Systems*, pp. 10169–10179, 2018.

[12] Denevi, G., Ciliberto, C., Grazzi, R., and Pontil, M. Learning-to-learn stochastic gradient descent with biased regularization. *arXiv preprint arXiv:1903.10399*, 2019.

[13] Denevi, G., Stamos, D., Ciliberto, C., and Pontil, M. Online-within-online meta-learning. In *Advances in Neural Information Processing Systems*, pp. 13110–13120, 2019.

[14] Denevi, G., Pontil, M., and Ciliberto, C. The advantage of conditional meta-learning for biased regularization and fine tuning. *Advances in Neural Information Processing Systems*, 33, 2020.

[15] Domke, J. Provable smoothness guarantees for black-box variational inference. Preprint arXiv:1901.08431, accepted for ICML 2020, 2019.

[16] Fallah, A., Mokhtari, A., and Ozdaglar, A. On the convergence theory of gradient-based model-agnostic meta-learning algorithms. In *International Conference on Artificial Intelligence and Statistics*, pp. 1082–1092, 2020.

[17] Finn, C., Rajeswaran, A., Kakade, S., and Levine, S. Online meta-learning. *arXiv preprint arXiv:1902.08438*, 2019.

[18] Hazan, E. Introduction to online convex optimization. *Foundations and Trends® in Optimization*, 2:157–325, 01 2016.

[19] Jose, S. T., Simeone, O., and Durisi, G. Transfer meta-learning: Information-theoretic bounds and information meta-risk minimization. *arXiv preprint arXiv:2011.02872*, 2020.

[20] Knoblauch, J., Jewson, J., and Damoulas, T. Generalized variational inference: Three arguments for deriving new posteriors. *arXiv preprint arXiv:1904.02063*, 2019.

[21] Konobeev, M., Kuzborskij, I., and Szepesvári, C. On optimality of meta-learning in fixed-design regression with weighted biased regularization. *arXiv preprint arXiv:2011.00344*, 2020.

[22] Lin, W., Khan, M. E., and Schmidt, M. Fast and simple natural-gradient variational inference with mixture of exponential-family approximations. In Chaudhuri, K. and Salakhutdinov, R. (eds.), *Proceedings of the 36th International Conference on Machine Learning*, volume 97 of *Proceedings of Machine Learning Research*, pp. 3992–4002, Long Beach, California, USA, 09–15 Jun 2019. PMLR.

[23] Maurer, A. Bounds for linear multi-task learning. *Journal of Machine Learning Research*, 7:117–139, 2006.

[24] Maurer, A., Pontil, M., and Romera-Paredes, B. The benefit of multitask representation learning. *Journal of Machine Learning Research*, 17(81): 1–32, 2016.

[25] Meunier, D. Meta-learning meets variational inference: Learning priors with guarantees. MSc thesis, Université Paris Saclay, 2020. `https://dimitri-meunier.github.io/files/RikenReport.pdf`.

[26] Nesterov, Y. *Introductory lectures on convex optimization: A basic course*, volume 87. Springer Science & Business Media, 2004.

[27] Orabona, F. A modern introduction to online learning. *arXiv preprint arXiv:1912.13213*, 2019.

[28] Parikh, N. and Boyd, S. Proximal algorithms. *Foundations and Trends® in optimization*, 1(3):127–239, 2014.

[29] Pentina, A. and Lampert, C. A PAC-Bayesian bound for lifelong learning. In *Proc. 31st International Conference on Machine Learning*, pp. 991–999, 2014.

[30] Romera-Paredes, B., Aung, H., Bianchi-Berthouze, ., and Pontil, M. Multilinear multitask learning. In *International Conference on Machine Learning*, pp. 1444–1452, 2013.

[31] Rothfuss, J., Fortuin, V., and Krause, A. Pacoh: Bayes-optimal meta-learning with pac-guarantees. *arXiv preprint arXiv:2002.05551*, 2020.

[32] Ruvolo, P. and Eaton, E. Ella: An efficient lifelong learning algorithm. In *Proc. 30th International Conference on Machine Learning*, pp. 507–515. 2013.

[33] Shalev-Shwartz, S. Online learning and online convex optimization. *Foundations and Trends® in Machine Learning*, 4(2):107–194, 2012.

[34] Thrun, S. and Pratt, L. *Learning to learn.* Kluwer Academic Publishers, 1998.

[35] Yamada, M., Koh, T., Iwata, T., Shawe-Taylor, J., and Kaski, S. Localized lasso for high-dimensional regression. In *Artificial Intelligence and Statistics*, pp. 325–333. PMLR, 2017.

[36] Zhou, P., Yuan, X., Xu, H., Yan, S., and Feng, J. Efficient meta learning via minibatch proximal update. In *Advances in Neural Information Processing Systems*, pp. 1534–1544, 2019.